Buckling of Cylinders with Cutouts

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A numerical solution is presented for the problem of stability of elastic cylindrical shells with rectangular cutouts. Reinforcements around the cutouts and other discrete stiffening elements are considered and the loading can be either axial or lateral. The analysis is based on a two-dimensional finite difference scheme, and thus the numerical solution entails treatment of a large system of nonlinear algebraic equations. A straightforward Newton-Raphson method requires an excessive computational effort, therefore, a modified Newton method is employed in which the iteration strategy is controlled internally during the computation. Simple experiments were performed for the verification of the analysis. The agreement between test and theory appears to be satisfactory. The effects of cutout dimensions and of reinforcement around the cutouts are demonstrated by numerical examples.

Nomenclature

a = radius
A_r,A_s = cross-sectional area of ring and stringer
A,B = linear and nonlinear operators [see Eq. (8)]
C_p = vector of constant terms [see Eq. (7)]
E,E_r,E_s = Young's modulus for shell, ring and stringer
F = polynomial operator [see Eq. (7)]
I_x,I_y,I_z = moments of inertia of ring and stringer
h = thickness of shell wall
J_r,J_s = torsional constants of ring and stringer
k = $h^2/12a^2$
l = length of cylinder
P = axial load on one fourth of the circumference
t = height of reinforcing frame
U,U_r,U_s = strain energy of shell, ring and stringer
V = total potential energy
W = work done by external forces
u,v,w = axial, circumferential, and radial displacement components
w_1,w_2,w_3 = displacements (see Fig. 2)
x,θ,z = axial, circumferential and radial coordinates
X = represents displacement components [see Eq. (7)]
$\bar{u},\bar{v},\bar{w},\xi$ = $(u,v,w,x)/a$
α,β = rotation of tangents to the midsurface
$\epsilon_\xi,\epsilon_\theta,\gamma_{\xi\theta}$ = strains at the middle surface
$\chi_\xi,\chi_\theta,\chi_{\xi\theta}$ = changes of curvature and twist
ν = Poisson's ratio
φ = angle covered by cutout (see Fig. 8)

Introduction

As a result of the spectacular improvements in computer technology, much progress has been made lately in the field of shell analysis. For example, computer programs are available for the linear as well as nonlinear prebuckling analysis and for the stability analysis of almost any shell of revolution with axisymmetric loading.[1] The shell of revolution is a common design element; however, in many practical applications the axial symmetry is destroyed by cutouts in the shell wall. For such cases the designer lacks adequate methods of analysis. In general, he designs the basic shell as if there were no cutouts and hopes that sturdy reinforcements at the cutout will prevent premature failure. For a linear analysis, computer programs are available for shells of rather general shape.[2,3] These programs are based on the finite element technique or on a two-dimensional finite difference analysis. In both cases it is impossible to order the unknowns in such a way that the coefficient matrix of the corresponding equation system is narrowly banded, as in the case of axisymmetric behavior. The computer time for solution of the equation system increases considerably as it is approximately proportional to the square of the bandwidth. The linear analysis may be useful in many applications but if the ultimate load or the collapse load of the shell is to be determined, nonlinear terms must be retained in the analysis. For such a case the linear equation system must be solved a large number of times. The load is increased stepwise until a limit point is reached. For each value of the load, the iteration scheme involves repeated solution of the linear system. Thus it is clearly impossible to obtain theoretical solutions for many of the problems which the designer will encounter. However, a computer program was derived here for the collapse of a cylindrical shell with rectangular cutouts. Application of this program indicates that there are practical cases for which solutions can be obtained. Also certain modifications such as variable mesh spacing will undoubtedly improve the program. The usefulness of the program can be further enhanced by engineering ingenuity in its application.

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Analysis

The stability limit of a shell may often be found by determination of a bifurcation point. For instance if shells of revolution are axisymmetrically loaded this approach is valid with either a linear or a nonlinear prebuckling analysis. The bifurcation analysis is strictly applicable only if the buckling pattern is orthogonal to the pattern of prebuckling deformations. As an approximation it has been applied, however, also to other cases such as cylindrical panels with nonuniform loading.[4] It seems likely that the bifurcation point for the cases treated in Ref. 4 represents reasonably well the load level at which the lateral displacement begins to grow at a rapidly increasing rate. However, this load may be well below the collapse load of the panel. To determine the critical load rigorously for such a case, a nonlinear analysis must be employed. For axially or laterally loaded cylinders with cutouts, it appears that significant redistribution of the stresses takes place before the collapse load is reached. Consequently a complete nonlinear analysis was used here for determination of the collapse load.

It was assumed that at least one plane of symmetry exists since the form of the equation system becomes somewhat more complicated if the complete cylinder is considered. When nonlinear terms are included in the analysis, bifurcation is possible in modes which are orthogonal to the prebuckling deformation pattern. Thus, when planes of symmetry exist, there is also a possibility of bifurcation into deformation patterns which are antisymmetric with respect to those planes.

The analysis was restricted to cylindrical panels whose boundaries follow coordinate lines. The numerical solution combines a variational approach and a two-dimensional finite difference approximation. This approach leads to a large equation system in which the unknowns are the displacement components at the mesh points. The total potential energy is expressed in terms of the displacement components and equilibrium configurations are found through minimization of this energy expression. Due to the use of the energy method only geometric boundary conditions need to be specified. Thus it is very easy to simulate the free edge conditions at cutouts in the shell. Also, the effects of any number of discrete stiffeners may easily be accounted for through inclusion of the energy of the stiffeners.

The equations on which the analysis is based are given below. The strain-displacement relations are

$$\begin{gathered}\epsilon_\theta = \bar{v}_{,\theta} + \bar{w} + \alpha^2/2,\ \epsilon_\xi = \bar{u}_{,\xi} + \beta^2/2 \\ \gamma_{\xi\theta} = \bar{u}_{,\theta} + \bar{v}_{,\xi} + \alpha\beta\end{gathered} \tag{1}$$

The rotations of the tangents to the middle surface are

$$\alpha = -\bar{w}_{,\theta} + \bar{v},\ \beta = -\bar{w}_{,\xi} \tag{2}$$

The changes of curvature and twist

$$\begin{gathered}\chi_\theta = \alpha_{,\theta} = -\bar{w}_{,\theta\theta} + \bar{v}_{,\theta} \\ \chi_\xi = \beta_{,\xi} = -\bar{w}_{,\xi\xi} \\ \chi_{\xi\theta} = (\alpha_{,\xi} + \beta_{,\theta})/2 = -\bar{w}_{,\xi\theta} + \bar{v}_{,\xi}/2\end{gathered} \tag{3}$$

The strain energy U of the cylinder is then given by

$$U = \frac{Eha^2}{2(1-\nu^2)} \int_0^{l/a} \int_0^{\theta_0} \left[\epsilon_\xi^2 + \epsilon_\theta^2 + 2\nu\epsilon_\xi\epsilon_\theta + \left(\frac{1-\nu}{2}\right) \times \gamma_{\xi\theta}^2 + k\{\chi_\xi^2 + \chi_\theta^2 + 2\nu\chi_\xi\chi_\theta + 2(1-\nu)\chi_{\xi\theta}^2\} \right] d\theta d\xi \tag{4}$$

The strain energy U_r of a ring and the strain energy U_s of a stringer may be expressed in a similar way. Eccentricity is not considered and it is assumed that the centroids and shear centers of the stiffeners coincide. Hence

$$\begin{gathered}U_r = \frac{E_r a}{2} \int_0^{\theta_0} \left[A_r\epsilon_\theta^2 + \frac{I_x}{a^2}\chi_\theta^2 + \frac{I_z}{a^2}(\bar{u}_{,\theta\theta} - \bar{w}_{,\xi})^2 + \frac{J_r}{2(1+\nu)a^2}\chi_{\xi\theta}^2 \right] d\theta \\ U_s = \frac{E_s a}{2} \int_0^{l/a} \left[A_s\epsilon_\xi^2 + \frac{I_y}{a^2}\chi_\xi^2 + \frac{I_z}{a^2}\bar{v}_{,\xi\xi}^2 + \frac{J_r}{2(1+\nu)a^2}\chi_{\xi\theta}^2 \right] d\xi\end{gathered} \tag{5}$$

the total potential energy then is

$$V = U + U_r + U_s + W \tag{6}$$

where W is the work done by the external forces. A uniform rectangular mesh is introduced and the derivatives of the displacement components are expressed in terms of the displacements at the mesh points by use of central difference approximations. From Eqs. (1–5), it is seen that only approximations up to second-order derivatives are needed. While the discrete values of the radial displacement component w are defined at the mesh points, the tangential components u, v are defined at points half-way between the mesh points. This procedure leads to more accurate results and to better numerical stability.

By use of these finite difference approximations the total potential energy is obtained as a fourth-order polynomial in the discrete displacement components. The minimization of the potential energy leads to a large algebraic equation system of the form

$$FX = C_p \tag{7}$$

where F is a polynomial operator of third degree, X is the vector of displacement components, and C_p is the vector of constant terms generated by the external forces in the pth load increment. The calculation of partial derivatives and the formulation of the operator F and the vector C_p are carried out by the computer during execution of each case.

For convenience, the operator F is expressed as

$$F = A + B \tag{8}$$

where A and B represent the linear and nonlinear parts of F, respectively. Iteration methods for the solution of Eq. (7) reduce the nonlinear problem to the solution of a series of linear equation systems. For example, there is the natural generalization of Newton's method given by

$$X_{n+1} = X_n - (F'X_n)^{-1}(FX_n - C_p) \tag{9}$$

where $F'X_n$ is a matrix representing the Frechet derivative of F evaluated at the displacement vector X_n.[5] Since F is a polynomial operator of third degree, the elements of $F'X_n$ are simply expressions of second order in the displacement components. A more general iteration scheme for solution of Eq. (7) is

$$(F'X_m)(X_{n+1} - X_n) = C_p - FX_n \tag{10}$$

This iteration formula contains, as special cases, the most commonly used iterative methods. The simplest of these methods is based on a rearrangement of Eq. (7)

$$AX_{n+1} = C_p - BX_n \tag{11}$$

This formula is given by the choice of $x_0 = 0$ and $m \equiv 0$ in Eq. (10) and corresponds to repeated solution of the linear equation system which is obtained by substitution of the previous iterate for the unknowns in the nonlinear terms. The method is often effective in cases for which the effects of nonlinear terms are small. Unfortunately for a cylinder with a cutout it appears that Eq. (11) often fails to converge well before the collapse load is reached.

If in Eq. (10) only one iteration is carried out for each load step, this step must be chosen small enough to guarantee accuracy. Such a method can be shown to be equivalent to the so called "incremental analysis," in which the load step is sufficiently small to allow omission of terms of higher order in the increments. This method, as well as the standard Newton method [Eq. (9)], requires the computation of the Frechet derivative matrix and the solution of the linear equation system for each iteration.

Since F was generated from a polynomial function, the matrix $F'X_m$ is symmetric. The solution of the equation system is obtained by a direct method based on factorization of $F'X_m$ into the product of a lower triangular and an upper triangular matrix. The factorization retains the original moderately banded form and in view of the symmetry only the lower triangle needs to be computed (this holds even if the matrix is not positive definite).

In the modified Newton method the factorization of $F'X_m$ is generally avoided through retention of the same matrix

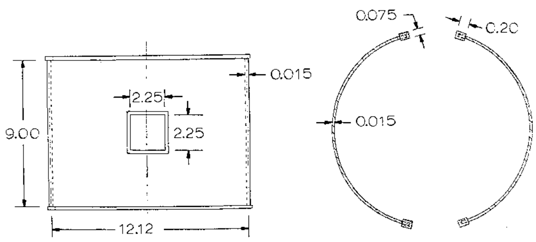

Fig. 1 Geometry of test specimen.

$F'X_m$ for a number of iterations, n, and load steps p. In this way each iteration requires much less computer time but the convergence is somewhat slower. As soon as convergence difficulties occur at a load step, the matrix is updated through substitution of the latest values of the unknowns, x_{p-1}, and the matrix is refactored. The modified Newton method was found to be the most economic and was thus used in the analysis.

Sometimes, as the collapse load is approached, for example, it will be necessary to modify the size of the increment in the external loads. An optimum fixed iteration strategy cannot be determined a priori and consequently the control of load step size and of refactoring is determined as much as possible by the computer during execution.

Experiment

An aluminum cylinder was available for testing. The cylinder, with a radius to thickness ratio of about 400 had previously been tested to failure under axial compression. Damage to the specimen was prevented by use of a deformation limiting mandrel, as was described in Ref. 6. The classical buckling load for the cylinder was about 8000 lb, but at experiment, presumably due to the effects of initial displacements, the cylinder buckled at 5200 lb. At this test as well as in the following with cutouts the cylinder edges were clamped.

Two symmetrically located rectangular holes were cut in the cylinder as shown in Fig. 1. First an axial load was applied to the cylinder with unreinforced cutout, and deformations around the cutout were recorded. The load was limited to 2500 lb so that the cylinder would not be damaged but could be used again in a test with a reinforcing frame applied around the cutout. The geometry of this frame is also shown in Fig. 1. After the reinforcement was introduced the cylinder was tested again. This time the test was continued to collapse, which occurred at a load of 4800 lb, only slightly below the critical load for a cylinder without cutout.

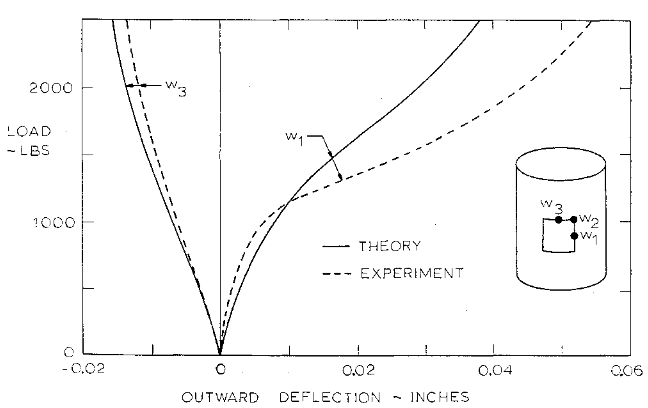

Fig. 2 Comparison of experimental and theoretical results.

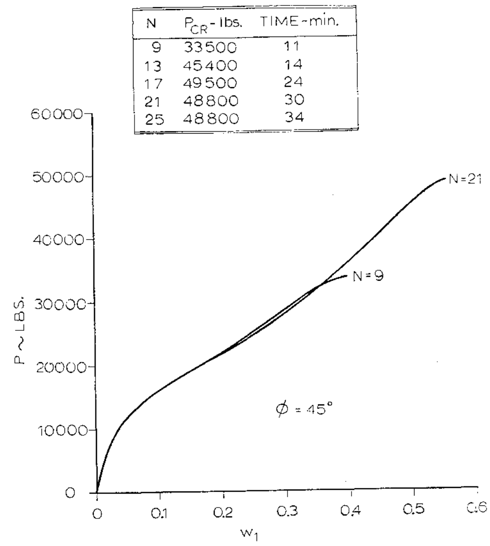

Fig. 3 Effect of mesh size for shells with unreinforced cutouts.

Results

For verification of the validity of the computer program it was first applied in a stability analysis for a complete cylinder (no cutouts) under external pressure. Nonsymmetric displacements were triggered in the analysis through inclusion of a small nonsymmetric pressure component. The critical pressure was found as the value of the uniform pressure at which the lateral displacements appear to approach an asymptotic value. The pressure so found agrees well with the critical external pressure as predicted by bifurcation theory.

The computer program was also applied in an analysis of the cylinder under axial compression that previously had been tested. For the cylinder with unreinforced cutout, Fig. 2 shows a comparison of the computed and measured displacements. The displacement component w_2 was approximately equal to zero in theory as well as in experiment. It appears that the theory predicts the general behavior of the shell reasonably well but that more accuracy is desirable. It was shown that considerably better results could be obtained if a finer mesh size was used in the analysis, but this would be quite expensive. For the cylinder with the reinforced cutout the displacements were extremely small in theory as well as in experiment. The theory indicated a collapse load of 6250 lb. This is 30% above the experimental load and it seems that the difference may be made consider-

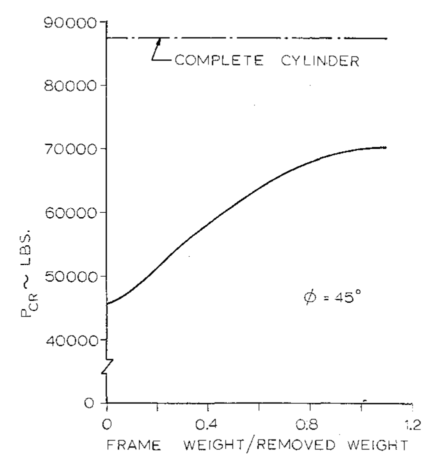

Fig. 4 Critical loads for shells with reinforcements.

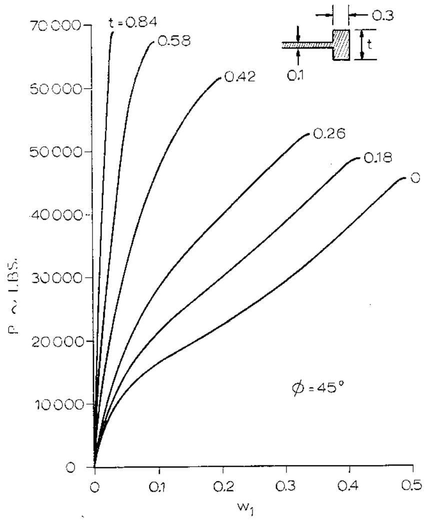

Fig. 5 Load displacement curves for shells with reinforcements.

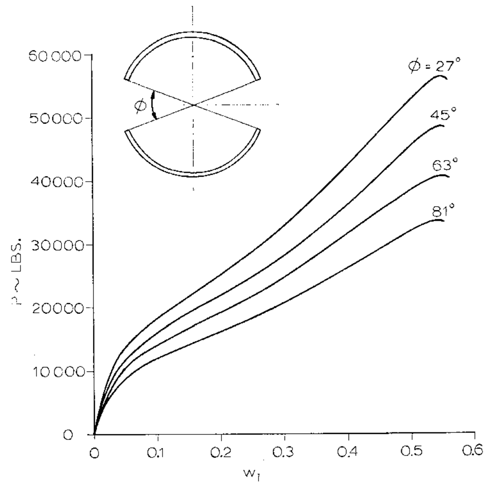

Fig. 7 Load displacement curves for shells with unreinforced cutouts.

ably smaller if a finer mesh could be used. It appears that the computer program can be considerably improved by use of a variable mesh spacing, and it seems reasonable to postpone further studies of this particular case until such a program is available. In the meantime it is possible to study, with the present program, the behavior of thicker cylinders, for which a coarser mesh can be used.

A parameter study was performed for cylinders with a radius to thickness ratio of 100. For all the cylinders in the study the radius was 10 in., the thickness 0.1 in., and the length 20 in. Young's modulus was 10^7 psi and Poisson's ratio 0.3. Only elastic behavior was considered. All cylinders had two diametrically opposite rectangular cutouts located at midlength. The cutouts covered 40% of the shell length.

First the influence of the mesh size was studied. Because of the symmetry conditions it was sufficient to study a cylindrical panel covering half the length and one quarter of the periphery. The number of mesh points in the axial direction was held constant (11 points). The number of points in the circumferential direction was varied from 9 to 25. The cylinders were loaded by axial compression, and as the end shortening was controlled, rather than the applied load, it was possible to carry the computations beyond the point of maximum load. The results are shown in Fig. 3. The nets with 25 and with 21 points lead to practically identical results. Reasonably close estimates are obtained for all the nets except the one with only 9 points. The displacement w_1 is shown as a function of the load for two cases. The computer times corresponding to each of the cases are also shown in the figure. Let n_x and n_θ designate the number of mesh stations in the axial and circumferential directions. The unknowns may be numbered such that the computational effort for solution of the equation system is either proportional to $n_x^3 n_\theta$ or to $n_x n_\theta^3$. If $n_\theta > n_x$ and the most economic choice is made, the computer time should be approximately proportional to n_θ. This seems to be confirmed by the execution times shown in Fig. 3.

A study was made also of the influence of reinforcements around the cutouts. In view of the results discussed previously it was deemed sufficient to use a mesh with 13 points in the circumferential direction. The reinforcements consisted of frames around the edge of the cutout. The width of the frame was held constant, 0.3 in. but its height was varied. The computer time was found to be much less for shells with reinforced cutouts. While the analysis of the unreinforced shell required a run time of 14 min, a reasonably close estimate could be obtained in 3 min for the collapse load of the cylinder with the heaviest frame. Figure 4 shows how the critical load varies with the height of the reinforcing frame. Load displacement curves are shown in Fig. 5 for a series of cylinders with frames of different sizes. Sometimes it is assumed in design, as a rule of thumb, that the critical load for the cylinder with a cutout equals that of the complete cylinder if the volume of the reinforcement is the same as that of the removed material. Therefore the critical load was plotted here as a function of the weight of the frame

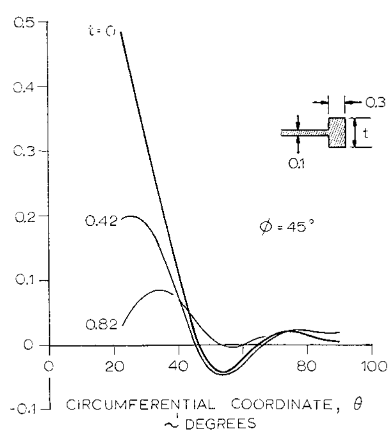

Fig. 6 Deformation at maximum load for shells with reinforcements.

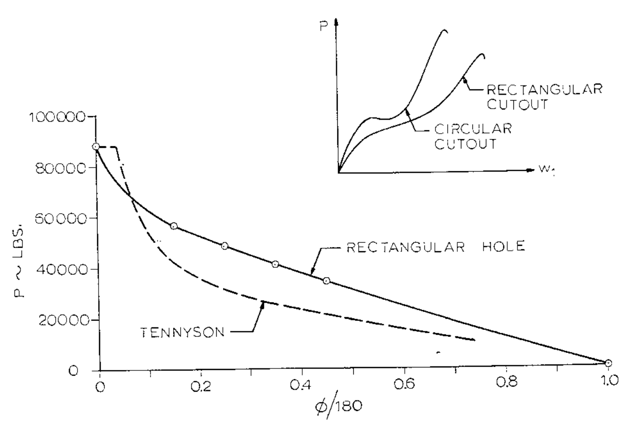

Fig. 8 Critical loads for shells with unreinforced cutouts.

normalized with respect to the weight of material removed. Figure 6 shows for some of these cases how the midlength displacements, at maximum load, vary with the circumferential coordinate.

Finally a study was made of the influence of the size of the cutouts. Only the circumferential dimension was varied. The smallest cutout covered an angle φ of 45° and for the largest φ was 81°. In this study a mesh was used with 21 points in the circumferential direction. Load displacement (w_1) curves are shown in Fig. 7 for four cases. In addition to collapse load for these four values of the hole size, it is easy to find the critical load for $\varphi = 0$ and, of course, it must be zero for $\varphi = \pi$. By use of these values a curve was drawn in Fig. 8, showing the collapse load as a function of the hole size. The broken curve in the figure represents the results indicated by Tennyson's[7] experiments on cylinders with circular cutouts. It seems somewhat surprising that the results for circular holes should be lower than for rectangular holes. However, it appears possible that for shells with circular cutouts the primary buckling mode is local and that the axial load can be considerably increased before total collapse occurs. This primary local buckling does not occur for shells with rectangular cutouts because the redistribution of stresses during loading is more drastic for these shells. In Fig. 8 it is also indicated how the load displacement curves corresponding to the different types of cutouts would differ in order to explain such a behavior.

Conclusion

For reasons of economy the parameter study just discussed had to be quite limited. It does still give some indication about how the critical load is affected by the presence of a cutout and how the detremental effects can be alleviated by use of reinforcements. A more important result is that convergence is obtained with decreasing mesh size for the relatively thick walled cylinders and that the agreement with experimental results is reasonably good for the thinner cylinders. Thus, the feasibility of a nonlinear analysis of shells of a rather general shape is established.

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